Question: (A) How many leaves does a full m-ary tree with L levels have?

(B) How many edges does it have?

height $= 2 = h$

eg This is a 3-level ($h=3$) 3-ary tree

In this example, there are 9 leaves, 12 edges.

Guesses.

(A) $m(h+1) = mL$, $\boxed{m^h}$

(B) $\displaystyle\sum_{n=1}^{h} m^n$

leaves of h height.

(A) When $h=0$, (# leaves) $= 0 = l(h)=0$, $l(1)=m$

If we know $l(h)$, how can we get $l(h+1)$.

$$\therefore\ l(h+1) = m\,l(h)$$

recursive formula for $l(h)$.

$$\begin{cases} l(1) = m & \leftarrow \text{m leaves for height 1} \\ l(h+1) = m\,l(h) \end{cases}$$

$$l(2) = m\,l(1) = m\cdot m = m^2$$

$$l(3) = m\,l(2) = m\cdot m^2 = m^3$$

$$\vdots$$

$$\boxed{l(h) = m^h} \quad \text{for } h \geq 1 \qquad l(0)=0.$$

$E(h)$ = (leaves of $h=1$) + (leaves of $h=2$) + ... + (leaves of h)

edges at height h $\quad m + m^2 + \cdots + \cdots + m^h$

mult by m

m.

$$E(h) = \sum_{k=1}^{h} m^k = \sum_{n=0}^{h-1} m^{n+1} \qquad m^{n+1} = m \cdot m^n$$

Geometric series formula

$$S = \sum_{n=0}^{N} a \cdot r^n = \boxed{a + ar + ar^2 + \dots + ar^N}$$

$$rS = ar + ar^2 + ar^3 + \dots + ar^{N+1}$$

subtract $(1-r)S = a - ar^{N+1}$

$$\Rightarrow \text{Sum} = S = \frac{a - ar^{N+1}}{1-r} \qquad r \neq 1$$

$a = 1^{st}$ term, $r =$ ratio

$N+1 =$ # of terms.

$a = m^1 = m$, $r = m$, $N+1 = h$

$$\Rightarrow E(h) = \frac{a - ar^{N+1}}{1-r} = \frac{m - m \cdot m^h}{1-m} \qquad m \geq 2$$

$$= \frac{m(1-m^h)}{1-m} = \boxed{\frac{m(m^h-1)}{m-1}}$$

As a result of these computations

For any (not necessarily full) m-ary tree of height h,

$$\left.\begin{aligned} \text{\# of leaves} &= l(h) \leq m^h \\ \text{\# of edges} &= E(h) \leq \frac{m(m^h-1)}{m-1} \end{aligned}\right\} \text{ for } m \geq 2.$$

This means # of vertices $= V(h) \leq \dfrac{m(m^h-1)}{m-1} + 1$.

Eg for binary trees, $m=2$

$$\Rightarrow \ell(h) \leq 2^h$$

$$E(h) \leq \frac{2(2^h-1)}{2-1} = 2(2^h-1) = \boxed{2^{h+1}-2}.$$

$$V(h) \leq 2^{h+1}-1.$$

Note: # of leaves in m-ary tree of height h is $\ell(h) \leq m^h$

i.e. $\ell \leq m^h$.

→ let's "solve" for h: We'll use the function

$f(x) = \log_m(x)$

increasing function

means if $x \leq y$ then $f(x) \leq f(y)$.

→ apply $\log_m(x)$ to both sides.

$$\log_m(\ell) \leq \log_m(m^h)$$

$$\Rightarrow \boxed{h \geq \log_m(\ell)}$$

Since h is an integer,

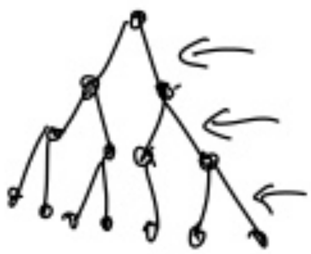
$$h \geq \lceil \log_m(\ell) \rceil$$

"ceiling of $\log_m(\ell)$" ← least integer ≥ the number.

$\lceil 3.8 \rceil = 4$ $\lceil 14.1 \rceil = 15$ $\lceil -0.8 \rceil = 0$.

For example: If you want to design a chess tournament with single elimination with a pair playing each other on each day, with 300 contestants, you would need

$\lceil \log_2(300) \rceil$ days



(binary tree: count leaves = 300)

$2^8 = 256 \quad 2^9 = 512$

$\log_2(300) = 8.$ something.

$\Rightarrow \lceil \log_2(300) \rceil = \boxed{9 \text{ days}}$ ..